THE UK UNIVERSITY INTEGRATION BEE 2022/23

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ROUND ONE SOLUTIONS

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Sponsored by

1. $\int_0^1 \frac{1}{\sqrt{x-x^2}}\mathrm{d}x$

Solution 1

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x-x^2}}\mathrm{d}x &= \int_0^1 \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)}}\mathrm{d}x \\ &= \left[\arcsin\left(2\left(x-\frac{1}{2}\right)\right)\right]_0^1 = \frac{\pi}{2} - -\frac{\pi}{2} = \pi\end{aligned}$$

Solution 2

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt{x-x^2}} &= \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\mathrm{d}x \\ &= B\left(\frac{1}{2},\frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \\ &= \pi\end{aligned}$$

2. $\int_0^{100} \lceil x \rceil \lfloor x \rfloor \mathrm{d}x$, where $\lfloor x \rfloor$&$\lceil x \rceil$ are the greatest integer less than x and the smallest integer greater than x, respectively.

$$\begin{aligned}
\int_0^{100} \lceil x \rceil \lfloor x \rfloor \mathrm{d}x &= \sum_{n=1}^{100} \int_{n-1}^{n} n(n-1)\mathrm{d}x \\
&= \sum_{n=1}^{100} n(n-1)\,(n-(n-1)) = \sum_{n=1}^{100} n^2 - n \\
&= \frac{(2*100+1)(100+1)(100)}{6} - \frac{100(100+1)}{2} \\
&= 67 \times 101 \times 50 - 50 \times 101 = 66 \times 101 \times 50 = 6666 \times 50 = 333300
\end{aligned}$$

3. $\displaystyle\int_0^{\pi} \cos(x + \cos(x))\mathrm{d}x$

$$\text{By making use of the substitution } u = \pi - x,$$
$$\int_0^{\pi} \cos(x + \cos(x))\mathrm{d}x = -\int_0^{\pi} \cos(x + \cos(x))\mathrm{d}x$$
$$\text{Hence, } \int_0^{\pi} \cos(x + \cos(x))\mathrm{d}x = 0$$

4. $\int_0^1 \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}\mathrm{d}x$
We begin by finding an alternate expression for the integrand

$$y=\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}=\sqrt{x+y}$$

Rearranging the above equation to make y the subject yields

$$y=\sqrt{x+\frac{1}{4}}+\frac{1}{2}$$

Thus, we may substitute this expression into the question

$$\begin{aligned}\int_0^1 \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}\mathrm{d}x &= \int_0^1 \sqrt{x+\frac{1}{4}}+\frac{1}{2}\mathrm{d}x \\ &= \left[\frac{2}{3}\left(x+\frac{1}{4}\right)^{\frac{3}{2}}+\frac{1}{2}x\right]_0^1 = \frac{2}{3}\sqrt{\frac{5}{4}}^3+\frac{1}{2}-\sqrt{\frac{1}{4}} \\ &= \frac{\sqrt{5}^3}{12}+\frac{5}{12}=\frac{5}{12}\left(\sqrt{5}+1\right)\end{aligned}$$

5. $\int_0^1 \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \mathrm{d}x$

$$\begin{aligned}
\int_0^1 \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \mathrm{d}x &= \lim_{k\to\infty} \sum_{n=1}^{k} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x} - n\mathrm{d}x \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} [\ln x]_{\frac{1}{n+1}}^{\frac{1}{n}} - n\left(\frac{1}{n}\frac{1}{n+1}\right) \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \ln(n+1) - \ln(n) - \frac{1}{n+1} \\
&= \lim_{k\to\infty} \ln(k+1) - \sum_{n=1}^{k} \frac{1}{n+1} \\
&= \lim_{k\to\infty} \ln(k+1) + 1 - \sum_{n=1}^{k} \frac{1}{n} = 1 - \gamma
\end{aligned}$$

(where γ is the Euler Mascheroni constant)

6. $\int_0^1 \frac{\arctan x + \operatorname{arccot} x}{x^2+1} \mathrm{d}x$

$$\begin{aligned}\int_0^1 \frac{\arctan x + \operatorname{arccot} x}{x^2+1} \mathrm{d}x &= \frac{\pi}{2} \int_0^1 \frac{1}{x^2+1} \mathrm{d}x \\ &= \frac{\pi^2}{8}\end{aligned}$$

Where the first equality uses the formula $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$, which can easily be shown by considering a right triangle with perpendicular sides $1, x$.

7. $\displaystyle\int_0^{\frac{\pi}{2}} x \prod_{i=1}^{\infty} \cos\left(\frac{x}{2^i}\right) \mathrm{d}x$

Consider the following identity $\cos(x) = \dfrac{\sin(2x)}{2\sin(x)}$. Now we will evaluate the product

$$\begin{aligned}
\prod_{i=1}^{\infty} \cos\left(\frac{x}{2^i}\right) &= \lim_{k\to\infty} \prod_{i=1}^{k} \cos\left(\frac{x}{2^i}\right) \\
&= \lim_{k\to\infty} \prod_{i=1}^{k} \frac{\sin\left(\frac{x}{2^{i-1}}\right)}{2\sin\left(\frac{x}{2^i}\right)} \\
&= \lim_{k\to\infty} \frac{1}{2^k} \prod_{i=1}^{k} \frac{\sin\left(\frac{x}{2^{i-1}}\right)}{\sin\left(\frac{x}{2^i}\right)} \text{ this is a telescoping product} \\
&= \frac{\sin x}{\lim_{k\to\infty} 2^k \sin\left(\frac{x}{2^k}\right)} = \frac{\sin x}{x}
\end{aligned}$$

Now we may evaluate the integral

$$\int_0^{\frac{\pi}{2}} x \prod_{i=1}^{\infty} \cos\left(\frac{x}{2^i}\right) \mathrm{d}x = \int_0^{\frac{\pi}{2}} x \frac{\sin x}{x} \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x \mathrm{d}x = 1$$

8. $\displaystyle\int_0^{\frac{\pi}{4}} \ln(\cot x - 1)\mathrm{d}x$

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \ln(\cot x - 1)\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln(1 - \tan x)\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln(\tan x)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(1 - \tan x)\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln\left(\tan\left(\frac{\pi}{4} - x\right)\right)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(1 - \tan x)\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln(1 - \tan x)\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\mathrm{d}x
\end{aligned}
$$

Let $I = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\mathrm{d}x$, then

$$\begin{aligned} I = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}x\right)\right)\mathrm{d}x \\ &= \int_0^{\frac{\pi}{4}} \ln 2\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\mathrm{d}x \\ &= \frac{\pi}{4}\ln 2 - I \end{aligned}$$

Hence, $I = \frac{\pi}{8}\ln 2$

9. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(b\sin x)}{\sin x}\mathrm{d}x$

$$\text{Let } I(b) = \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(b\sin x)}{\sin x}\mathrm{d}x$$

Differentiate both sides with respect to b.

$$\begin{aligned}
I'(b) &= \int_0^{\frac{\pi}{2}} \frac{1}{1+b^2\sin^2 x}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{1+b^2\cos^2 x}\mathrm{d}x \text{ (the method is similar for } \sin) \\
&= 2\int_0^{\frac{\pi}{2}} \frac{1}{2+b^2+b^2\cos(2x)}\mathrm{d}x
\end{aligned}$$

Let $t = \tan x$

$$\begin{aligned}
I'(b) &= 2\int_0^{\infty} \frac{1}{2+b^2+b^2\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{\mathrm{d}t}{1+t^2} \\
&= \int_0^{\infty} \frac{\mathrm{d}t}{1+b^2+t^2} \\
&= \frac{1}{\sqrt{1+b^2}}\left[\arctan\left(\frac{t}{\sqrt{1+b^2}}\right)\right]_0^{\infty} \\
&= \frac{\pi}{2\sqrt{1+b^2}}
\end{aligned}$$

Now we may integrate to obtain $I(b)$

$$I(b) - I(0) = I(b) = \frac{\pi}{2}\int_0^b \frac{\mathrm{d}b}{\sqrt{1+b^2}} = \frac{\pi}{2}\sinh^{-1} b$$

10. $\displaystyle\int_0^\infty \frac{x^3}{e^x+1}\mathrm{d}x$

$$\begin{aligned}\int_0^\infty \frac{x^3}{e^x+1}\mathrm{d}x &= \int_0^\infty \frac{x^3e^{-x}}{1+e^{-x}} \\ &= \int_0^\infty x^3e^{-x}\sum_{n=0}^{\infty}(-1)^n e^{-nx}\mathrm{d}x \\ &\text{(where we have used the infinite geometric series expansion)} \\ &= \sum_{n=0}^{\infty}(-1)^n\int_0^\infty x^3e^{-(1+n)x}\mathrm{d}x \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(4)}{(n+1)^4} \\ &= \eta(4)\Gamma(4) = \frac{7}{8}\zeta(4)\Gamma(4) = \frac{21}{4}\zeta(4) = \frac{7\pi^4}{120}\end{aligned}$$

Where η, ζ, Γ are the Dirichlet Eta, Riemann Zeta, and Gamma functions respectively. The relation between Dirichlet Eta and the Zeta function can be derived by considering odd and even terms of the series:

$$\eta(s) = (1-2^{1-s})\zeta(s)$$

11. $\displaystyle\int_0^{\frac{1}{4}} \sum_{n=0}^{\infty} \binom{2n}{n} x^n \mathrm{d}x$
Notice that the integrand resembles a Maclaurin series expansion. Recall the Maclaurin series expansion of $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

where $f^{(n)}(0)$ denotes the n^{th} derivative of f evaluated at $x = 0$.
We will force the integrand into the above form and try to find a closed form for the sum by inspection.

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{x^n}{n!}$$

Hence, we are looking for f such that $f^{(n)}(0) = \dfrac{(2n)!}{n!}$.
We can construct the following recurrence relation,

$$f^{(n)}(0) = \frac{(2n)(2n-1)}{n} f^{(n-1)}(0) = 2(2n-1) f^{(n-1)}(0)$$

At this point we can guess $f(x) = \dfrac{1}{\sqrt{1-4x}}$

$$\int_0^{\frac{1}{4}} \sum_{n=0}^{\infty} \binom{2n}{n} x^n \mathrm{d}x = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x}} \mathrm{d}x = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{4x}} \mathrm{d}x = \left[\sqrt{x}\right]_0^{\frac{1}{4}} = \frac{1}{2}$$

12. $\int_0^\infty \cos(x^2)\mathrm{d}x$
We will provide a solution by Laplace Transforms.

$$I(t) = \int_0^\infty \cos(tx^2)\mathrm{d}x$$

$$\begin{aligned} L\{I(t)\} &= \int_0^\infty \int_0^\infty \cos(tx^2)\mathrm{d}x e^{-st}\mathrm{d}t \\ &= \int_0^\infty \int_0^\infty \cos(tx^2)e^{-st}\mathrm{d}t\mathrm{d}x \\ &= \int_0^\infty \frac{s}{s^2+x^4}\mathrm{d}x \end{aligned}$$

This integral can be computed by using partial fraction decomposition on the factorisation, $x^4 + s^2 = (x^2 - \sqrt{2s}x + 1)(x^2 + \sqrt{2s} + 1)$. However, we will leave that as an exercise.

$$L\{I(t)\} = \frac{\pi}{2\sqrt{2s}}$$

For those familiar with Laplace Transforms, you will see that

$$I(t) = L^{-1}\{\frac{\pi}{2\sqrt{2s}}\} = \sqrt{\frac{\pi}{8t}}$$

Thus, $I(1) = \dfrac{\sqrt{\pi}}{2\sqrt{2}}$

13. $\displaystyle\int_0^\infty \frac{\ln x}{1-x^2}\mathrm{d}x$

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{1-x^2}\mathrm{d}x &= \int_0^1 \frac{\ln x}{1-x^2}\mathrm{d}x + \int_1^\infty \frac{\ln x}{1-x^2}\mathrm{d}x \\
&= 2\int_0^1 \frac{\ln x}{1-x^2}\mathrm{d}x \text{ (by using the substitution } u = \frac{1}{x} \text{ on the second integral)} \\
&= 2\int_0^1 \ln x \sum_{n=0}^{\infty} x^{2n}\mathrm{d}x \\
&= 2\sum_{n=0}^{\infty}\int_0^1 x^{2n}\ln x\mathrm{d}x \\
&= 2\sum_{n=0}^{\infty} -\frac{1}{(2n+1)^2} = -\frac{\pi^2}{4}
\end{aligned}$$

14. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)}{\cos^2 x} \mathrm{d}x$
We will use integration by parts

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\ln(\sin x)}{\cos^2 x} \mathrm{d}x &= [\tan x \ln(\sin x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \tan x \cot x \mathrm{d}x \\ &= -\int_0^{\frac{\pi}{2}} 1 \mathrm{d}x = -\frac{\pi}{2}\end{aligned}$$

15. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^2 x(1+\cos x)}{(1+\cos x+\sin x)^2}\mathrm{d}x$
We may re-express the denominator to arrive at the solution

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos^2 x(1+\cos x)}{(1+\cos x+\sin x)^2}\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x(1+\cos x)}{2(1+\cos x)(1+\sin x)}\mathrm{d}x \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2 x(1-\sin x)}{1-\sin^2 x}\mathrm{d}x \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1-\sin x\mathrm{d}x = \frac{\pi}{4}-\frac{1}{2} = \frac{1}{4}(\pi-2)\end{aligned}$$

16. $\displaystyle\int_1^\infty \frac{x - \lfloor x \rfloor}{x^2} \mathrm{d}x$

By use of $u = \dfrac{1}{x}$ substitution we can show this integral is equivalent to problem 5. Therefore the answer is $1 - \gamma$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos t}{\cosh t} \mathrm{d}t$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos t}{\cosh t} \mathrm{d}t &= 4 \int_{0}^{\infty} \frac{\cos t}{e^{t} + e - t} \mathrm{d}t \\
&= 4 \int_{0}^{\infty} \frac{\cos t e^{-t}}{1 + e^{-2t}} \mathrm{d}t \\
&= 4 \int_{0}^{\infty} \cos t e^{-t} \sum_{n=0}^{\infty} (-1)^{n} e^{-2nt} \mathrm{d}t \\
&= 4 \sum_{n=0}^{\infty} (-1)^{n} \int_{0}^{\infty} \cos t e^{-(2n+1)t} \mathrm{d}t \\
&= 4 \sum_{n=0}^{\infty} (-1)^{n} \frac{2n+1}{(2n+1)^{2} + 1} = \pi \operatorname{sech}\left(\frac{\pi}{2}\right)
\end{aligned}$$

It's pretty tough to recognise that series as sec - you can use contour integration to do it. The original integral can also be done with a rectangular contour.

18. $\displaystyle\int_0^\infty \frac{\ln(x+1)}{x^2+1}\mathrm{d}x$
We begin with the substitution $x = \tan\theta$

$$\begin{aligned}
\int_0^\infty \frac{\ln(x+1)}{x^2+1}\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \ln(\tan\theta+1)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin\theta+\cos\theta) - \ln(\cos\theta)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\right)\mathrm{d}\theta + \frac{\pi}{2}\ln 2 \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \ln\left(\sin\left(\theta+\frac{\pi}{2}\right)\right)\mathrm{d}\theta + \frac{3\pi}{4}\ln 2 \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \ln\cos\theta\mathrm{d}\theta + \frac{3\pi}{4}\ln 2 \\
&= 2\int_0^{\frac{\pi}{4}} \ln\cos\theta\mathrm{d}\theta + \frac{3\pi}{4}\ln 2
\end{aligned}$$

Let $A = \displaystyle\int_0^{\frac{\pi}{4}} \ln\cos\theta\mathrm{d}\theta$ and $B = \displaystyle\int_0^{\frac{\pi}{4}} \ln\sin\theta\mathrm{d}\theta$

$$\begin{aligned}
A+B &= \int_0^{\frac{\pi}{4}} \ln\sin\theta\cos\theta\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln\sin 2\theta\mathrm{d}\theta - \frac{\pi}{4}\ln 2 \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin\theta\mathrm{d}\theta - \frac{\pi}{4}\ln 2 = -\frac{\pi}{4}\ln 2 - \frac{\pi}{4}\ln 2 = -\frac{\pi}{2}\ln 2 \\
B-A &= \int_0^{\frac{\pi}{4}} \ln\tan\theta\mathrm{d}\theta \\
&= \int_0^1 \frac{\ln x}{1+x^2}\mathrm{d}x \\
&= \int_0^1 \sum_{n=0}^{\infty}(-1)^n x^{2n}\ln x\mathrm{d}x \\
&= \sum_{n=0}^{\infty}(-1)^n \times \left(-\frac{1}{(2n+1)^2}\right) = -G
\end{aligned}$$

$$A = \frac{1}{2}((A+B)-(B-A)) = \frac{1}{2}\left(-\frac{\pi}{2}+G\right) = -\frac{\pi}{4}\ln 2 + \frac{1}{2}G$$

$$\int_0^\infty \frac{\ln(x+1)}{x^2+1}\mathrm{d}x = 2A + \frac{3\pi}{4}\ln 2 = \frac{\pi}{4}\ln 2 + G$$

19. $\int_0^{\pi} \sec x \ln\left(1+\frac{\cos x}{3}\right)$

Let $I(a) = \int_0^{\pi} \sec x \ln(1 + a\cos x)\mathrm{d}x$ and differentiate with respect to a.

$$\begin{aligned}
I'(a) &= \int_0^{\pi} \frac{\mathrm{d}x}{1+a\cos x} \\
\text{Let } t = \tan\frac{x}{2} & \\
I'(a) &= \int_0^{\infty} \frac{1}{1+a\frac{1-t^2}{1+t^2}} \times \frac{2\mathrm{d}t}{1+t^2} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{1+a+(1-a)t^2} \\
&= \frac{2}{1-a}\int_0^{\infty} \frac{\mathrm{d}t}{\frac{1+a}{1-a}+t^2} \\
&= \frac{2}{1-a}\sqrt{\frac{1-a}{1+a}}\left[\arctan\left(\sqrt{\frac{1+a}{1-a}}t\right)\right]_0^{\infty} \\
&= \frac{\pi}{\sqrt{1-a^2}}
\end{aligned}$$

$$\int_0^{\pi} \sec x \ln\left(1+\frac{\cos x}{3}\right) = I\left(\frac{1}{3}\right) = I\left(\frac{1}{3}\right) - I(0) = \int_0^{\frac{1}{3}} \frac{\pi}{\sqrt{1-a^2}}\mathrm{d}a = \pi\arcsin\left(\frac{1}{3}\right)$$

20. $\displaystyle\int_0^1 \frac{\ln(1+x+x^2)}{x}\mathrm{d}x$

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x+x^2)}{x}\mathrm{d}x &= \int_0^1 \frac{\ln\left(\frac{1-x^3}{1-x}\right)}{x}\mathrm{d}x \\
&= \int_0^1 \frac{\ln(1-x^3)}{x}\mathrm{d}x - \int_0^1 \frac{\ln(1-x)}{x}\mathrm{d}x \\
&= \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty} -\frac{x^{3n}}{n}\mathrm{d}x - \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty} -\frac{x^{n}}{n}\mathrm{d}x \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{3n-1}\mathrm{d}x + \sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\mathrm{d}x \\
&= -\sum_{n=1}^{\infty}\frac{1}{3n^2} + \sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{\pi^2}{18} + \frac{\pi^2}{6} = \frac{\pi^2}{9}
\end{aligned}$$

21. $\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\ln x + 1}}{x^2} \mathrm{d}x$
Let $u = \ln(x) + 1$

$$\begin{aligned}
\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\ln x + 1}}{x^2} \mathrm{d}x &= \int_0^{\infty} \frac{\sqrt{u}}{(e^{u-1})^2} e^{u-1} \mathrm{d}u \\
&= e \int_0^{\infty} \frac{\sqrt{u}}{e^{-u}} \mathrm{d}u \\
&= e\Gamma\left(\frac{3}{2}\right) = e\frac{\sqrt{\pi}}{2}
\end{aligned}$$

The last equality may be derived by using the recurrence property of Γ and the substitution $t = u^2$.

Traditionally, problem 21 is my favourite problem because there's a clever idea involved. Here, the alternative solution is a pretty clever idea. Write $1 = \ln e$ and then

$$\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\ln x + 1}}{x^2} \mathrm{d}x = \int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\ln(xe)}}{x^2} \mathrm{d}x$$

The lower bound being $\frac{1}{e}$ is the motivation for this; now we can substitute $u = ex$ and then $t = \ln u$, we get

$$\int_{\frac{1}{e}}^{\infty} \frac{\sqrt{\ln(xe)}}{x^2} \mathrm{d}x = e \int_1^{\infty} \frac{\sqrt{\ln u}}{u^2} \mathrm{d}u = e \int_0^{\infty} \sqrt{t} e^{-t} \mathrm{d}t = \frac{e\sqrt{\pi}}{2}$$

22. $\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right) \mathrm{d}x$

$$\begin{aligned}
\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right) \mathrm{d}x &= \int_0^\infty \ln\left(\frac{1+e^{-x}}{1-e^{-x}}\right) \mathrm{d}x \\
&= \int_0^\infty \ln(1+e^{-x})\mathrm{d}x - \int_0^\infty \ln(1-e^{-x})\mathrm{d}x \\
&= \int_0^\infty \sum_{n=1}^\infty \frac{(-1)^{n+1}e^{-nx}}{n}\mathrm{d}x + \int_0^\infty \sum_{n=1}^\infty \frac{e^{-nx}}{n}\mathrm{d}x \\
&= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} + \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{\pi^2}{4}
\end{aligned}$$

23. $\int_0^1 \sqrt[4]{\frac{1}{x} - 1} \mathrm{d}x$

$$\begin{aligned}\int_0^1 \sqrt[4]{\frac{1}{x} - 1} \mathrm{d}x &= B\left(\frac{5}{4}, \frac{3}{4}\right) \\ &= \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}+\frac{3}{4}\right)} = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{4\sin\left(\frac{\pi}{4}\right)} = \frac{\pi}{2\sqrt{2}}\end{aligned}$$

This can also be done by recognising the inverse of the integrand is $\frac{1}{x^4+1}$ which can be done via partial fractions & Sophie Germain identity, the beta function or contour integration.

24. $\displaystyle\int_0^{2\pi} e^{\cos x}\cos(nx - \sin x)\,\mathrm{d}x, n \in \mathbb{Z}$
Recognise, by taking the real part of Euler's forumula, that

$$\mathrm{Re}\left(e^{i\theta}\right) = \cos\theta$$

So

$$\begin{aligned}
\int_0^{2\pi} e^{\cos x}\cos(nx - \sin x)\mathrm{d}x &= \mathrm{Re}\left(\int_0^{2\pi} e^{\cos x} e^{i(nx-\sin x)}\mathrm{d}x\right) \\
&= \mathrm{Re}\left(\int_0^{2\pi} e^{inx+\cos x - i\sin x}\mathrm{d}x\right) \\
&= \mathrm{Re}\left(\int_0^{2\pi} e^{inx} e^{e^{-ix}}\mathrm{d}x\right) \\
&= \mathrm{Re}\left(\int_0^{2\pi} e^{inx}\sum_{k=0}^{\infty}\frac{e^{-ikx}}{k!}\mathrm{d}x\right) \\
&= \mathrm{Re}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^{2\pi} e^{i(n-k)x}\mathrm{d}x\right) \\
&= \mathrm{Re}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\times(2\pi\delta_{n,k})\right) = \frac{2\pi}{n!}
\end{aligned}$$

25. $\int_0^\infty \frac{\ln x \sin x}{x} \mathrm{d}x$
We will make use of the following formula for $\log x$ (a Frullani integral)

$$\log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} \mathrm{d}t$$

$$\begin{aligned}
\int_0^\infty \frac{\ln x \sin x}{x} \mathrm{d}x &= \int_0^\infty \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} \frac{\sin x}{x} \mathrm{d}t\mathrm{d}x \\
&= \int_0^\infty \frac{1}{t} \int_0^\infty \frac{\sin x}{x} \left(e^{-t} - e^{-xt}\right) \mathrm{d}x\mathrm{d}t \\
&= \int_0^\infty \frac{1}{t} \left(\frac{\pi}{2} e^{-t} - \frac{\pi}{2} + \arctan t\right) \mathrm{d}t \\
&= \left[\log t \left(\frac{\pi}{2} e^{-t} - \frac{\pi}{2} + \arctan t\right)\right]_0^\infty - \int_0^\infty \log t \left(-\frac{\pi}{2} e^{-t} + \frac{1}{1+t^2}\right) \mathrm{d}t \\
&= \frac{\pi}{2} \int_0^\infty \log t e^{-t} \mathrm{d}t = \frac{-\gamma\pi}{2}
\end{aligned}$$

26. $\displaystyle\int_0^1 \frac{x-1}{(x+1)\ln x}\mathrm{d}x$

Let $I(a) = \displaystyle\int_0^1 \frac{x^a-1}{(x+1)\ln x}\mathrm{d}x$ and differentiate with respect to a.

$$\begin{aligned} I'(a) &= \int_0^1 \frac{x^a}{x+1}\mathrm{d}x \\ &= \int_0^1 x^a \sum_{n=0}^{\infty}(-1)^n x^n \mathrm{d}x \\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{a+n}\mathrm{d}x \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{a+n+1} \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{x-1}{(x+1)\ln x}\mathrm{d}x &= I(1) = I(1) - I(0) \\ &= \int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n}{a+n+1}\mathrm{d}a \\ &= \sum_{n=0}^{\infty}(-1)^n \left[\ln(a+n+1)\right]_0^1 \\ &= \sum_{n=0}^{\infty}(-1)^n \ln(n+2) - (-1)^n \ln(n+1) \\ &= \sum_{k=0}^{\infty}\ln(2n+2) - \ln(2n+3) + \ln(2n+2) - \ln(2n+1) \\ &= \ln\left(\prod_{n=0}^{\infty}\frac{(2n+2)(2n+2)}{(2n+3)(2n+1)}\right) = \ln\left(\frac{\pi}{2}\right) \end{aligned}$$

where the last line uses the Wallis Product.

27. $\displaystyle\int_0^1 \frac{\sin(\log x) - \log x}{\log^2 x} \mathrm{d}x$
Let $u = -\log x$

$$\begin{aligned}
\int_0^1 \frac{\sin(\log x) - \log x}{\log^2 x} \mathrm{d}x &= \int_0^\infty \frac{u - \sin u}{u^2} e^{-u} \mathrm{d}u \\
&= \int_0^\infty \int_0^\infty (u - \sin u) e^{-u} t e^{-ut} \mathrm{d}t \mathrm{d}u \\
&= \int_0^\infty t \int_0^\infty (u - \sin u) e^{-(1+t)u} \mathrm{d}u \mathrm{d}t \\
&= \int_0^\infty t \left(\frac{1}{(1+t)^2} - \frac{1}{(1+t)^2 + 1} \right) \mathrm{d}t \\
&= \left[\ln(1+t) + \frac{1}{1+t} - \frac{1}{2} \ln\left((1+t)^2 + 1 \right) + \arctan(1+t) \right]_0^\infty \\
&= \left[\ln\left(\frac{1+t}{\sqrt{(1+t)^2 + 1}} \right) + \frac{1}{1+t} + \arctan(1+t) \right]_0^\infty \\
&= \frac{\ln 2}{2} + \frac{\pi}{4} - 1
\end{aligned}$$

28. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}} \ln\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}\right) \mathrm{d}x$

We start with the substitution $x = \cos\theta$

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1-x^2}} \ln\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}\right) \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sqrt{1+\cos\theta}+1}{\sqrt{1+\cos\theta}-1}\right) \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\frac{(\sqrt{1+\cos\theta}+1)^2}{(\sqrt{1+\cos\theta}-1)(\sqrt{1+\cos\theta}+1)}\right) \mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{2}} \ln\left(\sqrt{1+\cos\theta}+1\right) \mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \ln\cos\theta \mathrm{d}\theta
\end{aligned}$$

Making use of the identity $1+\cos x = \cos^2\left(\frac{x}{2}\right)$ and the substitution $t = \frac{\theta}{2}$

$$\begin{aligned}
&= 4\int_0^{\frac{\pi}{4}} \ln\left(1+\sqrt{2}\cos t\right) \mathrm{d}\theta + \frac{\pi}{2}\ln 2 \\
&= 4\frac{\pi}{4}\ln\sqrt{2} + 4\int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{\sqrt{2}}+\cos t\right) \mathrm{d}t + \frac{\pi}{2}\ln 2
\end{aligned}$$

Taking $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and using the sum to product identities

$$\begin{aligned}
&= \pi\ln 2 + 4\int_0^{\frac{\pi}{4}} \ln 2\mathrm{d}t \\
&+ 4\int_0^{\frac{\pi}{4}} \ln\left(\cos\left(\frac{t}{2}+\frac{\pi}{8}\right)\cos\left(\frac{t}{2}-\frac{\pi}{8}\right)\right) \mathrm{d}t \\
&= 2\pi\ln 2 + 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\cos\left(\frac{t}{2}\right)\right) \mathrm{d}t + 4\int_{-\frac{\pi}{4}}^{0} \ln\left(\cos\left(\frac{t}{2}\right)\right) \mathrm{d}t \\
&= 2\pi\ln 2 + \int_0^{\frac{\pi}{2}} \ln\cos\frac{t}{2}\mathrm{d}t
\end{aligned}$$

Using the fourier series expansion $\ln\cos\frac{x}{2} = -\ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n}\cos nx$

$$\begin{aligned}
&= 2\pi\ln 2 + \int_0^{\frac{\pi}{2}} -\ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n}\cos nt\mathrm{d}t \\
&= 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 4G
\end{aligned}$$

This problem can be solved without knowledge of the log cosine Fourier series in the same way as Q18.

29. $\displaystyle\int_0^{\frac{1}{2}} \ln(\sqrt{1-x} - \sqrt{x})\mathrm{d}x$

We will use the substitution $x = \sin^2\theta$

$$\begin{aligned}
\int_0^{\frac{1}{2}} \ln(\sqrt{1-x} - \sqrt{x})\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln\left(\cos\theta - \sin\theta\right) \times 2\sin\theta\cos\theta\mathrm{d}\theta \\
&= -\frac{1}{2}\left[\cos 2\theta \ln\left(\cos\theta - \sin\theta\right)\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\cos 2x\left(-\sin x - \cos x\right)}{\cos x - \sin x}\mathrm{d}x \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\sin x + \cos x\right)^2 \mathrm{d}x \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sin^2\left(x + \frac{\pi}{4}\right)\mathrm{d}x \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos\left(2x - \frac{\pi}{2}\right)\mathrm{d}x \\
&= -\left(\frac{\pi}{8} + \frac{1}{4}\right) = -\frac{\pi + 2}{8}
\end{aligned}$$

30. $\displaystyle\int_0^\infty \frac{\arctan x \ln(1+x^2)}{x(a^2+x^2)}\mathrm{d}x$

Unfortunately, we are yet to think of a solution that does not make use of complex analysis

Using the modulus argument form we have $1+ix = \sqrt{1+x^2}e^{\arctan x}$, then we may rewrite

$$\arctan x = \frac{i}{2}\left(\ln(1-ix) - \ln(1+ix)\right)$$
$$\ln\left(1+x^2\right) = \ln\left(1+ix\right) + \ln\left(1-ix\right)$$

Thus we get the following

$$\int_0^\infty \frac{\arctan x \ln(1+x^2)}{x(a^2+x^2)}\mathrm{d}x = \frac{i}{4}\int_{-\infty}^\infty \frac{\left(\ln(1-ix)-\ln(1+ix)\right)\left(\ln\left(1+ix\right)+\ln\left(1-ix\right)\right)}{x\left(a^2+x^2\right)}\mathrm{d}x$$

Where we have also used the "even"ness of the function to extend the limits of integration

$$= \frac{i}{4}\left(\int_{-\infty}^\infty \frac{\ln^2\left(1-ix\right)}{x\left(a^2+x^2\right)}\mathrm{d}x - \int_{-\infty}^\infty \frac{\ln^2\left(1+ix\right)}{x\left(a^2+x^2\right)}\mathrm{d}x\right)$$

To perform integration, we close the contour in the complex half-plane by the half-circle of the radius $R \to \infty$: for the first term - in the upper half-plane (counter-clockwise); for the second - in the lower (clockwise). In both cases logarithms do not have branch points in the designated closed contours. It is straightforward to show that the integrals along the big half-circles $\to 0$. $z=0$ is a removable singular point; therefore

$$\int_0^\infty \frac{\arctan x \ln(1+x^2)}{x(a^2+x^2)}\mathrm{d}x = 2\pi i\frac{i}{4}\operatorname*{Res}_{z=ia}\frac{\ln^2\left(1-ix\right)}{x\left(a^2+x^2\right)} - 2\pi i\left(-\frac{i}{4}\right)\operatorname*{Res}_{z=-ia}\frac{\ln^2\left(1+ix\right)}{x\left(a^2+x^2\right)} = \frac{\pi\ln^2\left(1+a\right)}{2a^2}$$